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Distance-Biregular Graphs with 2-Valent Vertices and Distance-Regular Line Graphs*

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A graph G is called *distance-regularized* if each vertex of G admits an intersection array. It is known that every distance-regularized graph is either distance-regular (DR) or *distance-biregular* (DBR). Note that DBR means that the graph is bipartite and the vertices in the same color class have the same intersection array. A (k, g)-*graph* is a k-regular graph with girth g and with the minimum possible number of vertices consistent with these properties. Biggs proved that, if the line graph $L(G)$ is distance-transitive, then G is either $K_{1,n}$ or a (k, g)-graph. This result is generalized to DR graphs by showing that the following are equivalent: (1) $L(G)$ is DR and $G \neq K_{1,n}$ for $n \geqslant 2$, (2) G and $L(G)$ are both DR, (3) subdivision graph $S(G)$ is DBR, and (4) G is a (k, g)-graph. This result is used to show that a graph S is a DBR graph with 2-valent vertices iff $S = K_{2,r}$ or S is the subdivision graph of a (k, g)-graph. Let $G^{(2)}$ be the graph with vertex set that of G and two vertices adjacent if at distance two in G. It is shown that for a DBR graph G, $G^{(2)}$ is two DR graphs. It is proved that a DR graph H without triangles can be obtained as a component of $G^{(2)}$ if and only if it is a (k, g)-graph with $g \geqslant 4$. © 1985 Academic Press, Inc.

1. INTRODUCTION AND DEFINITIONS

By a graph we mean a finite, undirected graph which has no loops and no multiple edges. Let G be a graph. By $V(G)$ and $E(G)$ we denote the vertex set and the edge set of G, respectively. By $d(u, v)$ we denote the usual

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distance in G between vertices u and v. For $v \in V(G)$ and $i \in \mathbb{N}$, $G_i(v)$ denotes the set of vertices at distance i from v. A vertex $u \in V(G)$ is said to be *distance-regularized* if for each $v \in V(G)$, the numbers

$$a_i(u) := |G_i(u) \cap G_1(v)|,$$

$$b_i(u) := |G_{i+1}(u) \cap G_1(v)|,$$

and

$$c_i(u) := |G_{i-1}(u) \cap G_1(v)|$$

depend only on the distance $d(u, v) = i$ and are independent of the choice of $v \in G_i(u)$. Let d be the diameter of G, and let u be a distance-regularized vertex of G. Then the array

$$\begin{bmatrix} * & c_1(u) & \cdots & c_{d-1}(u) & c_d(u) \\ 0 & a_1(u) & \cdots & a_{d-1}(u) & a_d(u) \\ b_0(u) & b_1(u) & \cdots & b_{d-1}(u) & * \end{bmatrix}$$

is called the *intersection array* for u. We shall call a connected graph in which every vertex is distance-regularized a *distance-regularized* graph. A special case of such graphs are the much studied *distance-regular* (DR) graphs in which all vertices have the same intersection array. Other examples are bipartite graphs in which the vertices in the same color partition have the same intersection array. We call these graphs *distance-biregular* (DBR). The complete bipartite graph $K_{m,n}$ is an example of a DBR graph. Some other less trivial examples can be found in [3]. In [3] it is proved that every distance-regularized graph is either DR or DBR.

In the present paper we shall be much concerned with DBR graphs. Therefore we introduce the following standardized notation. Let G be a DBR graph. Sets A and B denote the color partition of $V(G)$, u is a vertex in A and has intersection array

$$\begin{bmatrix} * & 1 & c_2 & \cdots & c_d \\ 0 & 0 & 0 & \cdots & 0 \\ r & b_1 & b_2 & \cdots & * \end{bmatrix}$$

or just

$$\begin{bmatrix} * & 1 & c_2 & \cdots & c_d \\ r & b_1 & b_2 & \cdots & * \end{bmatrix};$$

v is a vertex in B and has intersection array

$$\begin{bmatrix} * & 1 & f_2 & \cdots & f_d \\ s & e_1 & e_2 & \cdots & * \end{bmatrix}.$$

Note that $\deg(u) = r$ and $\deg(v) = s$.

In Section 2 of this paper we examine DBR graphs with vertices of degree 2 ($s = 2$). It is shown that these are either complete bipartite graphs $K_{2,r}$ or subdivision graphs of DR graphs. In Section 3 we characterize which DR graphs have a DBR subdivision graph. It turns out that these are precisely the (k, g)-graphs (see the definition below). We note that this result holds also for infinite locally finite graphs. The only such graphs which are not finite are regular trees which are (k, ∞)-graphs.

For $k \geqslant 1$ and $g \geqslant 3$ we define

$$\begin{aligned} n_0(k, g) &:= 1 + k + k(k-1) + \cdots + k(k-1)^{i-2} + k(k-1)^{i-1}, \quad g \text{ is odd}, \\ &:= 1 + k + k(k-1) + \cdots + k(k-1)^{i-2} + (k-1)^{i-1}, \quad g \text{ is even}, \end{aligned}$$

where $i = \lfloor g/2 \rfloor$. A *(k, g)-graph* is a k-regular graph with girth g and $n_0(k, g)$ vertices. The definition makes sense also for $g = \infty$. For each $k > 1$, a (k, ∞)-graph is the infinite k-regular tree. The class of (k, g)-graphs is very limited. The following complete list is known (cf. [2]):

(1) $k = 2$, $g \geqslant 3$: cycles of length g

(2) $k \geqslant 3$, $g = 3$: complete graphs K_{k+1}

(3) $k \geqslant 3$, $g = 4$: complete bipartite graphs $K_{k,k}$

(4) $k = 3$, $g = 5$: the Petersen graph

(5) $k = 7$, $g = 5$: the Hoffman–Singleton graph

(6) $k = 57$, $g = 5$: the existence of a $(57, 5)$-graph is neither proved nor disproved

(7) $k \geqslant 3$, $g = 6$, 8, or 12: for some values of k (not yet completely classified) a (k, g)-graph exists

(8) $k = 1$, $g = \infty$: the complete graph K_2

(9) $k \geqslant 2$, $g = \infty$: infinite k-regular trees.

In [2] Biggs proved the following result. If G has no vertices of degree 1 and $L(G)$ is distance transitive, then G is a (k, g)-graph. Our Proposition 3.1 generalizes this result to arbitrary DR graphs. Namely, the property that $L(G)$ is distance transitive can be weakened to $L(G)$ being DR, and this still implies that G is a (k, g)-graph. We note that Theorem 3.4 is even stronger. An immediate consequence to Theorem 3.4 is the classification of the DBR graphs with 2-valent vertices (cf. Corollary 3.5).

For a graph G let $G^{(2)}$ be the graph with vertex set $V(G)$ and two vertices adjacent if at distance two in G. We will show in Proposition 2.1 that for G a DBR graph, $G^{(2)}$ is the disjoint union of two DR graphs. It is an open question which DR graphs can be obtained in this way. In Corollary 3.6 we characterize which DR graphs without triangles have this property. It turns out that these are precisely the (k, g)-graphs for $g \geqslant 4$.

2. Distance Biregular Graphs with 2-Valent Vertices

2.1. Proposition. *Let G be a DBR graph. Then $G^{(2)}$ is the disjoint union of two DR graphs whose intersection arrays can be calculated from the arrays of G.*

Proof. Clearly, $G^{(2)}$ consists of two disjoint connected graphs, one having vertex set A, and the other having B as the vertex set. Assume that the arrays of G are in standardized form. Let $u \in A$ and consider $G_j^{(2)}(u) = G_{2j}(u)$. Note that this set is contained in A. Pick $x \in G_{2j}(u)$. In G, x is adjacent to no vertices in $G_{2j}(u)$ but is adjacent to c_{2j} vertices in $G_{2j-1}(u)$ and b_{2j} vertices in $G_{2j+1}(u)$. Each of the vertices in $G_{2j-1}(u)$ is adjacent to $b_{2j-1} - 1$ vertices in $G_{2j}(u)$ other than x. Similarly, the vertices in $G_{2j+1}(u)$ are adjacent to $c_{2j+1} - 1$ vertices in $G_{2j}(u)$ other than x.

Let $a_j^* := |G_{2j}(u) \cap G_2(x)|$. Then in G each of these a_j^* vertices in the intersection is at distance two from x and so has c_2 common neighbours with x. Hence counting edges in G between $G_1(x)$ and $G_{2j}(u) \cap G_2(x)$ in two ways we have:

$$a_j^* \cdot c_2 = c_{2j}(b_{2j-1} - 1) + b_{2j}(c_{2j+1} - 1),$$

giving

$$a_j^* = (c_{2j}(b_{2j-1} - 1) + b_{2j}(c_{2j+1} - 1))/c_2,$$

which is independent of the choice of x in $G_j^{(2)}(u)$.

Now by a similar argument we obtain

$$c_j^* = (c_{2j} \cdot c_{2j+1})/c_2 \qquad \text{and} \qquad b_j^* = b_{2j} \cdot b_{2j+1}/c_2$$

for the numbers of vertices adjacent to x in $G_{j-1}^{(2)}(u)$ and in $G_{j+1}^{(2)}(u)$, both independent of the vertex x. The intersection array for u in the component of $G^{(2)}$ on the vertex set A is thus:

$$\begin{bmatrix} * & 1 & c_2^* & c_3^* & \cdots \\ 0 & a_1^* & a_2^* & a_3^* & \cdots \\ b_0^* & b_1^* & b_2^* & b_3^* & \cdots \end{bmatrix}.$$

This array is independent of the choice of u from A. Hence this graph is DR. A similar argument for a vertex v of B yields the array of the second distance regular graph. ∎

Given a DBR graph G we call the two DR graphs which are the components of $G^{(2)}$ the *derived DR graphs* of G.

The following theorem neatly classifies interesting DBR graphs which have 2-valent vertices, i.e., $s=2$, though in the next section we shall obtain a much more concrete and unexpected classification.

2.2. THEOREM. *Let G be a DBR graph and let H be the derived graph with vertex set A. Let H have intersection array*

$$\begin{bmatrix} * & c_1^* & c_2^* & \cdots & c_t^* \\ 0 & a_1^* & a_2^* & \cdots & a_t^* \\ b_0^* & b_1^* & b_2^* & \cdots & * \end{bmatrix}.$$

If $t = \mathrm{diam}(H)$ is greater than one, the following conditions are equivalent:

(1) G is the subdivision graph of H, $G = S(H)$,

(2) $s=2$, and

(3) $a_1^* = 0$.

Proof. $(3) \Rightarrow (2)$. Let $v \in B$ and $s = \deg(v) > 2$. Pick u_1, u_2, u_3 distinct neighbours of v. Then u_2 and u_3 belong to $G_2(u_1) = H_1(u_1)$. Since u_2 and u_3 are adjacent in H, this contradicts $a_1^* = 0$.

$(2) \Rightarrow (1)$. Let $s=2$ and $\mathrm{diam}(H) > 1$. We show that for any two adjacent vertices, u and u_1, of H there is precisely one vertex v which is in G adjacent to both of them (thus verifying that $G = S(H)$). If there is another vertex, say v_1, in $G_1(u) \cap G_1(u_1)$ then $G_1(v) \cap G_1(v_1) = \{u, u_1\}$, and thus $f_2 = 2$. But then $e_2 = 0$, and hence $\mathrm{diam}(G) \leqslant 3$. This is a contradiction with the assumption that $\mathrm{diam}(H) > 1$.

$(1) \Rightarrow (3)$. Let $G = S(H)$ and $a_1^* \neq 0$. Let u be an element of A and let v_1 be a vertex in G which subdivides an edge of H joining two vertices of $H_1(u)$. Since $\mathrm{diam}(H) > 1$, there is a vertex v_2 in G which subdivides an edge of H joining a vertex in $H_1(u)$ and a vertex in $H_2(u)$. Note that v_1,

$v_2 \in G_3(u)$ and that $|G_2(u) \cap G_1(v_1)| = 2$ and $|G_2(u) \cap G_1(v_2)| = 1$, thus leading to a contradiction with distance regularizability of u. ∎

2.3. COROLLARY. *Let G and H be as in Theorem 2.2. If* $\operatorname{diam}(H) > 1$ *and* $s = 2$, *then* $G^{(2)} = H \cup L(H)$.

Proof. By Theorem 2.2, $G = S(H)$. It is easy to see that the derived graphs of G are H and $L(H)$. ∎

Some examples of subdivision graphs which are DBR are easily found, e.g., $S(K_{r,r})$ and $S(K_{r+1})$. By Corollary 2.3, for $S(H)$ to be DBR it is necessary that the line graph $L(H)$ is DR. However, it is not at all obvious if this is also a sufficient condition. Another question prompted by Corollary 2.3 is: which DR graphs have DR line graphs? These questions will be answered in the next section.

Proposition 2.1 tells us that the derived graphs of a DBR graph are DR. However, it is not clear which DR graphs can be obtained in this way. The following corollary to Theorem 2.2 gives a partial answer to this question. We note that using the results of Section 3 a more sophisticated answer is obtained (cf. Corollary 3.6).

2.4. COROLLARY. *Let H be a DR graph which has no triangles. If $L(H)$ is not DR, then H is not a derived graph of a DBR graph.*

Proof. Note that the diameter of H is greater than one, since the only possible DR graph without triangles and with diameter one is K_2. But $L(K_2) = K_1$ is DR.

H has no triangles, hence $a_1^* = 0$. By Theorem 2.2, if H is a derived graph of a DBR graph G, then $G = S(H)$. By Corollary 2.3, $L(H)$ is also a derived graph of G and hence it must be DR. ∎

Finally we consider the case when $\operatorname{diam}(H) = 1$, i.e., H is a complete graph.

2.5. THEOREM. *Let G and H be as in Theorem 2.2, and suppose that $s = 2$ and* $\operatorname{diam}(H) = 1$. *Then either*

(a) $H = K_n$, $n \geqslant 3$, *and* $G = S(H)$, *or*

(b) $H = K_2$ *and* $G = K_{2,r}$ *for some* $r \geqslant 1$.

Proof. If $H = K_2$ then clearly $G = K_{2,r}$. If $H = K_n$, $n \geqslant 3$, then for a vertex $v \in B = V(G) \backslash V(H)$ only two of the vertices of H are adjacent to v. Hence at least one is at distance 3 from v and so $G_3(v) \neq \varnothing$. Applying the argument of implication $(2) \Rightarrow (1)$ in the proof of Theorem 2.2 we obtain $G = S(H)$. ∎

3. Distance-Regular Line Graphs

In Section 2 we have shown that a necessary condition for a graph H to be the derived graph on the vertex set A of the bipartition of a DBR graph G which has $s = 2$ is that H and $L(H)$ are both DR. In this section we show that this condition is also sufficient. The proof is given in two steps. First we characterize which graphs have DR line graphs, and then we show that the subdivision graph of these graphs is DBR. Some other results are also proved.

In [2] Biggs proved that, if the line graph $L(G)$ of G is distance transitive, then G is either $K_{1,n}$ or a (k, g)-graph. Since distance transitive graphs are a special case of DR graphs, the following proposition is an extension of Biggs' result. From now on, G is always a connected graph having at least one edge.

3.1. Proposition. *If the line graph $L(G)$ of a graph G is DR, then either $G = K_{1,n}$ or G is a (k, g)-graph.*

Before giving the proof of Proposition 3.1 we need a simple lemma.

3.2. Lemma. *If $L(G)$ is DR, then either G is regular of degree greater than one, or it is a star $K_{1,n}$, $n \geq 1$.*

Proof. Since $L(G)$ is regular, G is either regular or biregular (bipartite, vertices of the same color class having the same degree). Suppose that G is not regular. If it contains a vertex of degree 1, it must be $K_{1,n}$ for some n greater or equal to 1. Assume now that G has no monovalent vertices. Let e be any edge of G chosen so that its end-vertices a and b satisfy $2 \leq \deg(a) < \deg(b)$. We consider two cases.

(1) There are adjacent edges e' and e'' such that $e' \neq e$, $e'' \neq e$, e' is incident with a, and e'' is incident with b (e, e', and e'' thus form a triangle).

(2) There is an edge e' incident with a which is not incident with an edge e'' at b.

In both cases it is easy to see that in $L(G)$ the numbers $|L(G)_1(e) \cap L(G)_1(e')|$ and $|L(G)_1(e) \cap L(G)_1(e'')|$ are different, thus contradicting the distance regularity of $L(G)$. ∎

Proof of Proposition 3.1. If G contains vertices of degree 1, then G is a star $K_{1,n}$ as guaranteed by Lemma 3.2. Otherwise G is regular of degree greater than 1. The case when G contains no cycles is also trivial. It must be an infinite regular tree which is a (k, ∞)-graph.

The rest of the proof is divided into four steps. It is assumed that G is k-regular, k greater than 1, and that the girth g of G is finite. For each edge e

of G, denote by $L_i(e)$, $i=0, 1, 2,\ldots$, the set of edges which are at distance i from e in $L(G)$.

Step 1. *We claim* $\operatorname{diam}(G)=\lfloor g/2 \rfloor$. Denote by $i := \lfloor g/2 \rfloor$. Let u be an arbitrary vertex of G. We show that no vertex of G is more than i apart from u. Let $e \in E(G)$ be an edge which is incident with u. Since $L(G)$ is DR, it is easy to see that e must lie on a cycle C in G which is of length g. It is also clear that the first $i+1$ columns of the intersection array for e in $L(G)$ are

$$\begin{bmatrix} * & 1 & 1 & \cdots & 1 & c_i & \cdots \\ 0 & k-2 & k-2 & \cdots & k-2 & a_i & \cdots \\ 2k-2 & k-1 & k-1 & \cdots & k-1 & b_i & \cdots \end{bmatrix}.$$

Two cases will be treated separately.

(a) *g is odd.* This case is illustrated for $g=7$ by Fig. 1, where the edges of G are drawn dashed and the edges of $L(G)$ bold. Choosing an edge e' which lies on C and is at distance i from e we see that $a_i \geqslant k-2+1=k-1$, since e' is adjacent to an edge e'' on C which is also i apart from e. Suppose that in G there is a vertex w which is at dstance $i+1$ from u. Let the sequence of edges $e_0, e_1,\ldots, e_i$ be a path of length $i+1$ from u to w (see Fig. 2), and let v' be the common vertex of e_{i-1} and e_i. The edge e_i is in $L_i(e_0)$. Therefore e_i has a_i adjacent edges which also belong to $L_i(e_0)$. Since $a_i > k-2$, there is at least one edge e' in $L_i(e_0) \cap L_1(e_i)$ which is not incident with v' but is adjacent to e_i. Hence e' is incident with w. Let v'' be the other vertex of e'. To be at distance i from e_0 there are two possibilities. Either there is a path of length $i-1$ from u to v'' or there is a path of length $i-1$ from v to v''. The former case is impossible since w is in $G_{i+1}(u)$. But in the latter case we obtain at v a closed walk which is of length $i-1+1+i=2i=g-1$ which is also not possible since this gives a cycle shorter than the girth g.

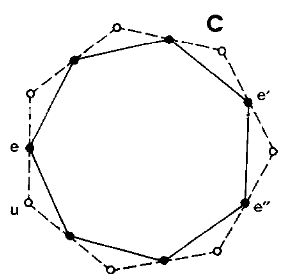

FIGURE 1

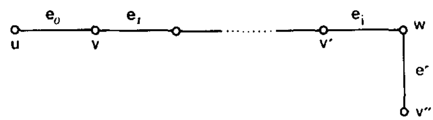

FIGURE 2

(b) *g is even.* See Fig. 3 where the case $g=8$ is illustrated. Let e' be the edge on C which is at distance i from e. If v_1 and v_2 are the vertices of e', no edge incident with either of them except the edges on C can be at distance less than i from e, since in such a case one would obtain in G a cycle of length less than $2i=g$. Therefore these edges are in $L_i(e)$, and consequently $c_i=2$, $a_i=2k-4$, and $b_i=0$. Suppose that the diameter of G is greater than i. Choose a vertex u in G such that $G_{i+1}(u)\neq\emptyset$. Let $e_0, e_1, ..., e_i$ be a path of length $i+1$ joining u with a vertex in $G_{i+1}(u)$. Note that $e_i \in L_i(e_0)$. Since $c_i=2$, there are exactly two edges (one of them is e_{i-1}) in $L_{i-1}(e_0)\cap L_1(e_i)$. As in the case (a) this leads to a contradiction. The details are left to the reader.

Step 2. *If g is even, then for each* $u\in V(G)$, *no two vertices in* $G_i(u)$ *are adjacent, where* $i=\lfloor g/2\rfloor$. Suppose that $v, w\in G_i(u)$ and that $e_i=(v, w)$ is an edge joining them. If $(u, u_1)=e_0, e_1, ..., e_{i-1}$ is a path of length i from u to v then $e_i\in L_i(e_0)$. Since $c_i=2$, there are exactly two edges in $L_{i-1}(e_0)\cap L_1(e_i)$. One of these edges is e_{i-1}. Let e' be the other edge and let $e_0=f_0, f_1, ..., f_{i-1}=e'$ be a path of length $i-1$ in $L(G)$. The edge e' is incident with v or w. In both cases f_1 is incident with u_1 as $d(u, v)=d(u, w)=i$. If e' is incident with v then the edges $f_1, f_2, ..., f_{i-1}$, e_i, $e_{i-1}, ..., e_2$, e_1 form a nontrivial closed walk of length $2i-2<g$, while if e' is incident with w, $f_1, f_2, ..., f_{i-1}$, $e_i, e_{i-1}, ..., e_2, e_1$ is a nontrivial closed walk of length $2i-1<g$. In both cases we have a shorter cycle than the girth of G, hence a contradiction.

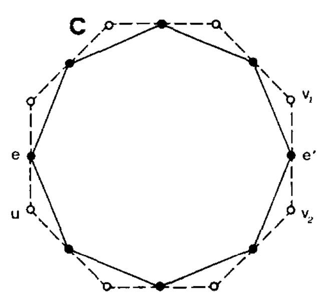

FIGURE 3

Step 3. *If g is odd, then for each $u \in V(G)$ and each $v \in G_i(u)$ there is a unique path of length i from u to v.* If there were two paths, one would obtain a cycle of length less than or equal to $2i < g$ which is impossible.

Step 4. *G is a (k, g)-graph.* Let $u \in V(G)$. For $1 \leqslant j < i$, $G_j(u)$ clearly contains $k \cdot (k-1)^{j-1}$ vertices. If g is even, then, by Step 2, $G_i(u)$ contains precisely k-times less vertices than $G_{i-1}(u)$, i.e., $(k-1)^{i-2}$. If g is odd, then by Step 3 $G_i(u)$ contains $k \cdot (k-1)^{i-1}$ vertices. In both cases G is a (k, g)-graph, since the number of its vertices is precisely $n_0(k, g)$.

The proof is completed. ▮

Note that each (k, g)-graph is DR. If g is even, it has the intersection array

$$\begin{bmatrix} * & 1 & 1 & \cdots & 1 & k \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ k & k-1 & k-1 & \cdots & k-1 & * \end{bmatrix}.$$

For odd g, the intersection array is

$$\begin{bmatrix} * & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & k-1 \\ k & k-1 & k-1 & \cdots & k-1 & * \end{bmatrix}.$$

3.3. Lemma. *If G is a (k, g)-graph, then the subdivision graph $S(G)$ is DBR.*

Proof. This is quite easy to see. Thus we leave all the details to the reader. Two cases must be considered.

(a) *g is odd.* The intersection array for $u \in V(G)$ in $S(G)$ is

$$\begin{bmatrix} * & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 2 \\ k & 1 & k-1 & 1 & k-1 & \cdots & 1 & k-1 & * \end{bmatrix}.$$

The array for $e \in E(G) \subseteq V(S(G))$ is

$$\begin{bmatrix} * & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 2 & 2 \\ 2 & k-1 & 1 & k-1 & 1 & \cdots & k-1 & 1 & k-2 & * \end{bmatrix}.$$

(b) *g is even.* For $u \in V(G)$ we have the intersection array in $S(G)$,

$$\begin{bmatrix} * & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & k \\ k & 1 & k-1 & 1 & k-1 & \cdots & k-1 & 1 & * \end{bmatrix}$$

and for $e \in E(G)$,

$$\begin{bmatrix} * & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 2 \\ 2 & k-1 & 1 & k-1 & 1 & \cdots & 1 & k-1 & * \end{bmatrix}. \blacksquare$$

3.4. THEOREM. *For a graph G the following conditions are equivalent:*

(1) *$L(G)$ is DR and $G \neq K_{1,n}$ for $n \geqslant 2$,*

(2) *G and $L(G)$ are both DR,*

(3) *$S(G)$ is a DBR graph, and*

(4) *G is a (k, g)-graph.*

Proof. (1) ⇒ (4). This is just Proposition 3.1.

(4) ⇒ (3). This is Lemma 3.3.

(3) ⇒ (2). The derived graphs of $S(G)$ are G and $L(G)$. These are both *DR* by Proposition 2.1.

(2) ⇒ (1). Obvious. ▮

The importance of the following two corollaries was discussed before.

3.5. COROLLARY. *A graph G with 2-valent vertices is DBR iff either $G = K_{2,r}$ or G is the subdivision graph of a (k, g)-graph.*

Proof. By Theorem 2.2 and Theorem 2.5 a DBR graph with vertices of degree 2 is either $K_{2,r}$ or the subdivision graph of some graph G'. By Theorem 3.4, G' is a (k, g)-graph. The converse is also immediate. $K_{2,r}$ is clearly DBR, and the subdivision graph of a (k, g)-graph is DBR by Lemma 3.3. ▮

3.6. COROLLARY. *Let H be a DR graph without triangles. Then H is a derived graph of a DBR graph iff H is a (k, g)-graph with $g \geqslant 4$.*

Proof. If H is a (k, g)-graph, it is a derived graph of $S(H)$. Conversely, if H is not a (k, g)-graph, its line graph is not DR, and by Corollary 2.4 H cannot be a derived graph of a DBR graph. ▮

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